JOURNAL OF DIFFERENTIAL EQUATIONS **140**, 186–208 (1997)
ARTICLE NO. DE973301

On Periodic Solutions of a Damped Wave Equation in a Thin Domain Using Degree Theoretic Methods*

Russell Johnson†

Facoltà di Ingegneria, Dipartimento di Sistemi e Informatica, Università degli Studi di Firenze, Via di S. Marta 3, 50139 Firenze, Italy

Paolo Nistri†

Facoltà di Ingegneria, Dipartimento di Sistemi e Informatica, Università degli Studi di Firenze, Via di S. Marta 3, 50139 Firenze, Italy

Mikhail Kamenski‡

Department of Mathematics, Voronezh State University, Universitetskaja p.1, Voronezh, Russia

Received January 17, 1997; revised March 26, 1997

We combine the methods of the topological degree with techniques developed by J. K. Hale and G. Raugel to study the periodic solutions of a damped nonlinear hyperbolic equation in a thin domain. © 1997 Academic Press

1. INTRODUCTION

The goal of this note is to discuss the solutions which are T-periodic with respect to t of the damped wave equation

$$\frac{\partial^2 u}{\partial t^2} = \Delta_X u + \frac{\partial^2 u}{\partial Y^2} - \beta \frac{\partial u}{\partial t} - \alpha u + g(t, X, Y, u), \tag{1}$$

when α and β are positive constants, g is an appropriate smooth function, and (X, Y) is a generic point of the "thin domain" $Q_\varepsilon = \Omega \times (0, \varepsilon) \subset \mathbf{R}^{N+1}$. Here Ω is a C^2-smooth bounded domain in $\mathbf{R}^N$ and ε is a small parameter.

* Research partially supported by the M.U.R.S.T., the C.N.R., and the R.F.F.I. Grant 96-01-00360.

† Fax: +39-55-4796363; e-mail: johnson(pnistri)@ingfil.ing.unifi.it.

‡ E-mail: mikhail@kam.vsu.ru.

0022-0396/97 $25.00

We will study the partial differential equation (1) when it is supplemented by the Neumann boundary condition

$$\frac{\partial u}{\partial \nu_\varepsilon} = 0 \qquad \text{on } \partial Q_\varepsilon. \tag{2}$$

Our interest will be in the existence of T-periodic solutions when g is T-periodic with respect to t. We will assume that an appropriate "reduced" problem at $\varepsilon = 0$ in the domain Ω admits an isolated T-periodic solution, and give conditions under which this solution continues to the problem (1)–(2) in Q_ε. Our method will be that of the topological degree for compact nonlinear operators. The application of the degree method is non-trivial because of the singularly perturbed nature of the problem (1)–(2).

The properties of the attractor A_ε defined by problem (1) has been considered by Hale and Raugel [5], when g does not depend on t, under Neumann and other boundary conditions. They are able to prove deep results concerning semicontinuity of A_ε as $\varepsilon \to 0$. In other papers, Hale and Raugel have given semicontinuity results for attractors in a broad range of problems defined on thin domains (e.g., [6], [7], [12]). See also Ciuperca [2], [3] for other results.

We will prove our results under the following hypotheses on the function g. First of all, g is assumed to be of class C^1 jointly in all its variables, and to be T-periodic in t: $g(t+T, X, Y, u) \equiv g(t, X, Y, u)$. Second, g satisfies the estimates (see lines (10)–(12) below)

$$|g_X(t, X, Y, u)| \leqslant a(1 + |u|^{\theta+1})$$

$$|g_Y(t, X, Y, u)| \leqslant a(1 + |u|^{\theta+1})$$

$$|g_u(t, X, Y, u)| \leqslant a(1 + |u|^{\theta})$$

for all values of its arguments t, X, Y, u. Here a is a suitable constant, and $\theta \in [0, \infty)$ if $N = 1$, $\theta \in [0, 2/(N-1))$ if $N > 1$.

We note explicitly that the growth rate θ is strictly less than the critical value $2/(N-1)$. This is because our methods require the validity of a Sobolev-type compact embedding result which is true only for $\theta < 2/(N-1)$. If $\theta = 2/(N-1)$, it is conceivable that one can study the existence of T-periodic solutions of problem (1) + (2) in Q_ε by using the theory of α-contractive semigroups. This question is currently under study. (We wish to thank the referee for the reference to the paper [1]).

We will make free use of the methods developed by Hale and Raugel in [5], and supplement them with those of the topological degree. Our results are not strictly comparable with theirs because we study only periodic solutions. The maximal attractor may of course have a complicated structure.

The note is organized as follows. In Section 2 we formulate and prove our main result. In Section 3 we illustrate it with an example.

2. THE MAIN RESULT

We proceed to study the problem (1)–(2) together with the limiting problem obtained in the following way by setting $\varepsilon = 0$. Write [5]

$$X = x, \qquad Y = \varepsilon y.$$

Then equation (1) becomes

$$\frac{\partial^2 u}{\partial t^2} = \Delta_x u + \frac{1}{\varepsilon^2}\frac{\partial^2 u}{\partial y^2} - \beta\frac{\partial u}{\partial t} - \alpha u + g(t, x, \varepsilon y, u), \tag{3}$$

and boundary condition (2) takes the form

$$\frac{\partial u}{\partial \nu} = 0 \qquad \text{on } \partial Q, \tag{4}$$

where $Q = \Omega \times (0, 1)$ and ν denotes the outward unit normal vector to Q. We suppose that Ω is a C^2-smooth domain, although our results could be proved if Ω is C^1 smooth at corners. (For this, one would use the methods developed in [5].)

Following [5], we introduce the following Banach spaces when $\varepsilon > 0$. Let X^1_ε be the space $H^1(Q)$ with the norm

$$\left(\|u\|^2_{1Q} + \frac{1}{\varepsilon^2}\left\|\frac{\partial u}{\partial y}\right\|^2_{0Q}\right)^{1/2}.$$

Here and below, $\|\cdot\|_{0Q}$ denotes the norm in $L^2(Q)$, and $\|\cdot\|_{jQ}$ that in $H^j(Q)$, $j = 1, 2, \ldots$ Further, let X^2_ε be the space

$$H^2(Q) \cap \left\{u \in H^2(Q) \,\middle|\, \frac{\partial u}{\partial \nu} = 0 \text{ in } \partial Q\right\}$$

with the norm

$$\left(\|u\|^2_{2Q} + \frac{1}{\varepsilon^2}\left\|\frac{\partial u}{\partial y}\right\|^2_{0Q} + \frac{1}{\varepsilon^2}\sum_{i=1}^{N}\left\|\frac{\partial^2 u}{\partial x_i \partial y}\right\|^2_{0Q} + \frac{1}{\varepsilon^4}\left\|\frac{\partial^2 u}{\partial y^2}\right\|^2_{0Q}\right)^{1/2}.$$

Let $U_\varepsilon(t)$ be the semigroup generated by the system of linear equations

$$\begin{aligned}\frac{\partial u}{\partial t} &= v\\ \frac{\partial v}{\partial t} &= \Delta_x u + \frac{1}{\varepsilon^2}\frac{\partial^2 u}{\partial y^2} - \beta v - \alpha u\end{aligned} \tag{5}$$

with the boundary condition (4). It is known [5, 7] that this semigroup is defined and C_0 in the space

$$Y^1_\varepsilon \triangleq X^1_\varepsilon \times X^0_\varepsilon \ni (u, v).$$

Here we have written X^0_ε for $L^2(Q)$. In the somewhat more general problem considered in [5], this space is defined in another way which is, however, in our situation equivalent to the space Y^1_ε as we have defined it.

The semigroup $U_\varepsilon(t)$ is also defined and of class C_0 in

$$Y^2_\varepsilon = X^2_\varepsilon \times X^1_\varepsilon \ni (u, v).$$

In both spaces, Y^1_ε and Y^2_ε, one has the exponential estimate [5]

$$\| U_\varepsilon(t)\|_{Y^j_\varepsilon \to Y^j_\varepsilon} \leqslant ce^{-\gamma t} \qquad (j=1, 2), \tag{6}$$

where c and γ are positive constants independent of ε.

We will need the following basic estimates (7)–(9); see [5, Lemma 3.1], [5, Lemma 4.1] for the proofs. If $u \in L^2(Q)$, define its projection

$$(Pu)(x) = \int_0^1 u(x, y)\, dy,$$

so that P maps $L^2(Q)$ to $L^2(\Omega)$. We have

$$\|Pu\|_{j, Q} \leqslant \|u\|_{j, Q} \qquad u \in H^j(Q), \qquad j = 0, 1, 2. \tag{7}$$

Furthermore

$$\|u - Pu\|_{X^0_\varepsilon} \leqslant c\varepsilon\, \|u\|_{X^1_\varepsilon}, \tag{8}$$

where we recall that $X^0_\varepsilon = L^2(Q)$. Finally, if $u \in H^2(Q)$ and $(\partial u/\partial y)(x, 0) = 0$, we have [5, p. 198]

$$\|u - Pu\|_{X^0_\varepsilon} + \varepsilon\, \|u - Pu\|_{X^1_\varepsilon} \leqslant c\varepsilon^2\, \|u\|_{X^2_\varepsilon}. \tag{9}$$

As above c denotes a positive constant independent of ε.

Next we discuss conditions on the function g. Our purpose is to study T-periodic solutions of the problem defined by (3) and (4). So we require that g be T-periodic with respect to t. We further assume that $g\colon \mathbf{R}\times\Omega\times[0,\varepsilon_0)\times\mathbf{R}\to\mathbf{R}$ is of class C^1 jointly in the variables t, x, Y and u for some $\varepsilon_0>0$, and that its derivatives satisfy the following estimates:

$$|g_x(t, x, Y, u)| \leqslant a(1+|u|^{\theta+1}) \tag{10}$$

$$|g_Y(t, x, Y, u)| \leqslant a(1+|u|^{\theta+1}) \tag{11}$$

$$|g_u(t, x, Y, u)| \leqslant a(1+|u|^{\theta}). \tag{12}$$

Here a is a positive constant, and θ is determined as follows: $\theta\in[0,\infty)$ if $N=1$, and $\theta\in[0,(2/(N-1)))$ for values $N\geqslant 2$. (Recall that $\dim Q = N+1$).

Let us discuss in more detail the concept of "T-periodic solution" which we will use below. Let $C_T(Y_\varepsilon^1)$ be the space of all continuous, T-periodic functions from $\mathbf{R}$ into Y_ε^1 with the usual norm:

$$\|w\| = \sup_{0\leqslant t\leqslant T} \|w(t)\|_{Y_\varepsilon^1}.$$

(In general, if E is a Banach space, we will use $C_T(E)$ to denote the space of continuous, T-periodic mappings $w\colon \mathbf{R}\to E$ with the norm $\|w\| = \sup_{0\leqslant t\leqslant T}\|w(t)\|_E$.) Define the following mapping F_ε on $C_T(Y_\varepsilon^1)$:

$$\begin{aligned} F_\varepsilon(w)(t) = U_\varepsilon(t)(I-U_\varepsilon(T))^{-1}\int_0^T U_\varepsilon(T-s)\, f_\varepsilon(w)(s)\, ds \\ +\int_0^t U_\varepsilon(t-s)\, f_\varepsilon(w)(s)\, ds. \end{aligned} \tag{13}$$

Here we have written $w(t) = \binom{u(t)}{v(t)} \in Y_\varepsilon^1$, and also

$$f_\varepsilon(w)(t, x, y) = \begin{pmatrix} 0 \\ g(t, x, \varepsilon y, u(t, x, y)) \end{pmatrix} \tag{14}$$

for the Nemytski operator generated by $\binom{0}{g}$. (We have abused notation; we should write $f_\varepsilon(w)(t)(x, y)$ in the right-hand side of (14)). It follows from estimate (6) with $j=1$ that the right-hand side of (13) is well-defined. We will see in a minute that F_ε actually maps $C_T(Y_\varepsilon^1)$ into itself.

Definition 1. By a "T-periodic solution of the problem (3)–(4)," we will mean a fixed point of the map F_ε.

The question of the exact relation between the set of fixed points of F_ε and the set of T-periodic distributional solutions of (3)–(4) has been studied in [10, 11]. It is known that a fixed point of F_ε is always a T-periodic distributional solution of (3)–(4).

We study the compactness properties of the operator F_ε. From the estimate (12), we have

$$|g(t, x, Y, u)| \leqslant \tilde{a}(1 + |u|^{\theta+1}).$$

For each $\varepsilon > 0$, there is a nonlinear Nemytski operator φ_ε on $C_T(L^p(Q))$, generated by g in the following way:

$$\varphi_\varepsilon(u)(t, x, y) = g(t, x, \varepsilon y, u(t, x, y)). \tag{15}$$

We choose $p \geqslant 2(\theta+1)$ if $N=1$, $p \in [2(\theta+1), (2N+2)/(N-1))$ if $N \geqslant 2$. Then φ_ε is well-defined on $C_T(L^p(Q))$, has values in $C_T(L^2(Q))$, is continuous, and is differentiable at each point $u \in C_T(L^p(Q))$. Moreover, if $\|u\|_{C_T(L^p(Q))} \leqslant R$, then

$$\|\varphi'_{\varepsilon, u}(u)\|_{C_T(L^2(Q))} \leqslant c(p, R), \tag{16}$$

where as indicated the constant depends on p and R (but not on ε). See [10].

From the embedding theorem of S. L. Sobolev one has that, for each fixed $\varepsilon > 0$, the map

$$\varphi_\varepsilon \colon C_T(X^1_\varepsilon) \to C_T(L^2(Q))$$

is well defined, satisfies a Lipschitz condition on every bounded set $\mathscr{M} \subset C_T(X^1_\varepsilon)$, and

$$\bigcup_{0 \leqslant t \leqslant T} \{\varphi_\varepsilon(u)(t) \mid u \in \mathscr{M}\} \subset L^2(Q)$$

is relatively compact. It follows that, for each $\varepsilon > 0$, the operator F_ε given by (13) maps $C_T(Y^1_\varepsilon)$ into itself and is compact (completely continuous). Thus we can hope to study the fixed points of F_ε by using the theory of the topological degree for compact operators.

The observations just made also show that the operator $f_\varepsilon \colon w = \binom{u}{v} \to \binom{0}{\varphi_\varepsilon(u)}$ maps $C_T(Y^1_\varepsilon)$ into itself.

Next we consider the semigroup $U_0(t)$ ($t \geqslant 0$) generated by the equations

$$\begin{aligned} \frac{\partial u}{\partial t} &= v \\ \frac{\partial v}{\partial t} &= \Delta_x u - \beta v - \alpha u \end{aligned} \tag{17}$$

with the Neumann boundary condition

$$\frac{\partial u}{\partial \nu} = 0 \qquad \text{in } \partial\Omega, \tag{18}$$

where ν is the unit outward normal to $\partial\Omega$. Let $\binom{u_0}{v_0}$ be an element of $H^1(\Omega) \times L^2(\Omega)$. Then $U_0(t)\binom{u_0}{v_0}$ is in $H^1(\Omega) \times L^2(\Omega)$, and one has the estimate

$$\| U_0(t) \|_{H^1(\Omega) \times L^2(\Omega) \to H^1(\Omega) \times L^2(\Omega)} \leqslant c e^{-\gamma t}, \tag{19}$$

where c and γ are positive constants.

For each fixed $\varepsilon > 0$, one has a natural inclusion $i: \Omega \to Q$ defined by $x \to (x, 0)$. The maps i induces an inclusion $\mathscr{J}: H^1(\Omega) \times L^2(\Omega) \to Y^1_\varepsilon$ via the formula $\mathscr{J}(u, v)(x, y) = (u(x), v(x))$ for each $(x, y) \in \Omega \times (0, 1)$. The map $\mathscr{J}$ is an isometry, and hence we can identify $U_0(t)(t)\binom{u_0}{v_0}$ with the element $\mathscr{J} U_0(t)\binom{u_0}{v_0}$ of Y^1_ε.

It is convenient to introduce some notation at this point. Denote by $J_\varepsilon: C_T(Y^1_\varepsilon) \to C_T(Y^1_\varepsilon)$ the operator defined as

$$J_\varepsilon w(t) = U_\varepsilon(t)(I - U_\varepsilon(T))^{-1} \int_0^T U_\varepsilon(T-s)\, w(s)\, ds + \int_0^t U_\varepsilon(t-s)\, w(s)\, ds.$$

Furthermore, let $f_\varepsilon: C_T(Y^1_\varepsilon) \to C_T(Y^1_\varepsilon)$ be the compact operator defined by (14). Then the map F_ε defined in (13) can be written as

$$F_\varepsilon(w) = J_\varepsilon f_\varepsilon(w).$$

We also define an operator J_0 acting on $C_T(H^1(\Omega) \times L^2(\Omega))$ as

$$J_0 w_0(t) = U_0(t)(I - U_0(T))^{-1} \int_0^T U_0(T-s)\, w_0(s)\, ds + \int_0^t U_0(t-s)\, w_0(s)\, ds.$$

Finally, we define the projection matrix

$$\mathbf{P} = \begin{pmatrix} P & 0 \\ 0 & P \end{pmatrix},$$

which maps $C_T(Y^1_\varepsilon)$ to $C_T(H^1(\Omega) \times L^2(\Omega))$ if interpreted in the obvious way.

With this notation, we define the operator F^0_ε by

$$F^0_\varepsilon(w) = \mathscr{J} J_0 \mathbf{P} f_\varepsilon(w), \tag{20}$$

which we can and will view as a map from $C_T(Y^1_\varepsilon)$ to itself. The complete continuity of this operator follows from the inequality (7). We note that the points in the image of the map F^0_ε belong to $\mathscr{J} C_T(H^1(\Omega) \times L^2(\Omega))$, where the notation has the obvious meaning. Thus by the restriction theorem [10] we have

$$\operatorname{ind}_{C_T(Y^1_\varepsilon)}(F^0_\varepsilon, \mathscr{U}) = \operatorname{ind}_{\mathscr{J} C_T(H^1(\Omega) \times L^2(\Omega))}(F^0_\varepsilon, \mathscr{U} \cap \mathscr{J} C_T(H^1(\Omega) \times L^2(\Omega))) \tag{21}$$

for each open set $\mathscr{U} \subset C_T(Y^1_\varepsilon)$ with the property that F^0_ε has no fixed points in $\partial \mathscr{U}$. (We recall that the index "ind" of a map F is related to the topological degree "deg" via the formula $\operatorname{ind}(F, \mathscr{U}) = \deg(I - F, \mathscr{U})$).

Let us remark that, if V is a bounded set in $C_T(H^1(\Omega) \times L^2(\Omega))$ and if $w = \binom{u}{v} \in V$, then

$$P\varphi_\varepsilon(u)(t) \underset{\varepsilon \to 0}{\overset{L^2(\Omega)}{\rightrightarrows}} \varphi_0(u)(t), \tag{22}$$

where $\varphi_0(u)(t, x) = g(t, x, 0, u(t, x))$, and where here and below we use the double arrow to indicate uniform convergence (in this case with respect to $t \in [0, T]$ and with respect to $u \in V_1 \triangleq \{u \in H^1(\Omega)$: there exists $v \in L^2(\Omega)$ such that $(u, v) \in V\}$).

To prove (22), we fix $p \geqslant 2(\theta + 1)$ if $N = 1$, $p \in [2(\theta + 1), (2N + 2)/(N - 1))$ if $N \geqslant 2$. Observe that, if $w \in V$ and $0 \leqslant t \leqslant T$, then $u(t) \in K$ where $K \subset L^p(\Omega)$ is a fixed compact set. We therefore have

$$\mu\{x \mid |u(t, x)| \geqslant k\} \underset{k \to +\infty}{\rightrightarrows} 0, \tag{23}$$

where μ denotes Lebesgue measure on Ω and the uniformity is with respect to $t \in [0, T]$ and $u \in V_1$. Define $\Omega_k = \{x \in \Omega \mid$ there exists $t \in [0, T]$ and $u \in V_1$ such that $|u(t, x)| \geqslant k\}$. We have

$$\begin{aligned}
&\int_\Omega \left| g(t, x, 0, u(t, x)) - \int_0^1 g(t, x, \varepsilon y, u(t, x))\, dy \right|^2 dx \\
&\quad \leqslant \int_{\Omega - \Omega_k} \left| \int_0^1 [g(t, x, 0, u(t, x)) - g(t, x, \varepsilon y, u(t, x))]\, dy \right|^2 dx \\
&\qquad + \int_{\Omega_k} \left| \int_0^1 [g(t, x, 0, u(t, x)) - g(t, x, \varepsilon y, u(t, x))]\, dy \right|^2 dx.
\end{aligned}$$

The first integral tends to zero as $\varepsilon \to 0$ because g is uniformly continuous on the set

$$[0, T] \times \bar{\Omega} \times [0, 1] \times [-k, k].$$

The second tends to zero as $\varepsilon \to 0$ by choice of p and the convergence (23). This proves (22).

We will use the convergence (22) to study the operator F_0 defined by the formula

$$F_0(w_0)(t) = U_0(t)(I - U_0(T))^{-1} \int_0^T U_0(T-s) \begin{pmatrix} 0 \\ \varphi_0(u_0)(s) \end{pmatrix} ds$$

$$+ \int_0^t U_0(t-s) \begin{pmatrix} 0 \\ \varphi_0(u_0)(s) \end{pmatrix} ds$$

for $w_0 = \binom{u_0}{v_0} \in C_T(H^1(\Omega) \times L^2(\Omega))$. Using the notation introduced above we can also write

$$F_0(w_0) = J_0 f_0(w_0),$$

where $f_0(w_0) = \binom{0}{\varphi_0(u_0)}$ and $\varphi_0(u_0)(t, x) = g(t, x, 0, u_0(t, x))$.

Proposition 1. *Let $V \subset C_T(H^1(\Omega) \times L^2(\Omega))$ be a bounded open set. If F_0 has no fixed points on the boundary of V, then for sufficiently small $\varepsilon > 0$ the operators F_ε^0 and $\mathscr{J} F_0 \mathbf{P}$ are homotopic on $\overline{\mathscr{J} V}$ (i.e., there is a homotopy between them which is fixed-point free on $\partial \mathscr{J} V$).*

Proof. Observe that $\partial(\mathscr{J} V)_{\mathscr{J} C_T(H^1(\Omega) \times L^2(\Omega))} = \mathscr{J}[(\partial V)_{C_T(H^1(\Omega) \times L^2(\Omega))}]$. We view F_0 as an operator on $\mathscr{J} C_T(H^1(\Omega) \times L^2(\Omega))$ and identify V with its image $\mathscr{J} V$ in $\mathscr{J} C_T(H^1(\Omega) \times L^2(\Omega))$.

Then F_0 has no fixed points on ∂V and so, by compactness, there is a constant $\nu > 0$ such that

$$\|w - F_0(w)\|_{C_T(H^1(\Omega) \times L^2(\Omega))} > \nu \qquad \text{if} \quad w \in \partial V \equiv \partial(\mathbf{P} \mathscr{J} V). \tag{24}$$

Now, using (22) and the exponential bound (19) on $U_0(t)$, we have

$$\|F_\varepsilon^0 \mathscr{J} w - F_0 w\|_{C_T(H^1(\Omega) \times L^2(\Omega))} \xrightarrow[\varepsilon \to 0]{} 0,$$

where the uniformity is with respect to $w \in V$. Therefore

$$\|\mathscr{J} w - \mathscr{J} F_0 \mathbf{P} \mathscr{J} w - \lambda(F_\varepsilon^0 \mathscr{J} w - \mathscr{J} F_0 \mathbf{P} \mathscr{J} w)\|_{\mathscr{J} C_T(H^1(\Omega) \times L^2(\Omega))} \geqslant \nu/2$$

for each $w \in \partial V$, if ε is sufficiently small. So $(1-\lambda)\,\mathscr{J} F_0 \mathbf{P} + \lambda F^0_\varepsilon$, $(0 \leqslant \lambda \leqslant 1)$ is a homotopy on $\mathscr{J} V$ with no fixed points on $\mathscr{J}(\partial V)$. This completes the proof of Proposition 1. ∎

Remark 1. If Z is a subset of a Banach space E, we let $B_r(Z)$ denote the "ball" of radius r about Z, that is

$$B_r(Z) = \{ w \in E \mid \operatorname{dist}(w, Z) < r \},$$

where as usual $\operatorname{dist}(w, z) = \inf\{ \|w - z\|_E \mid z \in Z \}$. We make the simple but important observation that, if $F: E \to E$ is a completely continuous operator, and if F has no fixed points on the boundary ∂Z of a bounded open set $Z \subset E$, then for sufficiently small $r > 0$, F has no fixed points on the boundary $\partial B_r(Z)$ of $B_r(Z)$, and

$$\operatorname{ind}_E(F, Z) = \operatorname{ind}_E(F, B_r(Z)).$$

PROPOSITION 2. *Suppose the hypotheses of Proposition* 1 *are satisfied. Then there exists* $r_0 > 0$ *such that, if* $0 < r \leqslant r_0$,

$$\begin{aligned} &\operatorname{ind}_{\mathscr{J} C_T(H^1(\Omega) \times L^2(\Omega))}(F^0_\varepsilon, B_r(\mathscr{J} V) \cap \mathscr{J} C_T(H^1(\Omega) \times L^2(\Omega))) \\ &\quad = \operatorname{ind}_{\mathscr{J} C_T(H^1(\Omega) \times L^2(\Omega))}(\mathscr{J} F_0 \mathbf{P}, \mathscr{J} V). \end{aligned}$$

Proof. We first remark that the inequality (24) is satisfied if $w \in \mathbf{P}[B_r(\mathscr{J} \partial V) \cap \mathscr{J} C_T(H^1(\Omega) \times L^2(\Omega))]$, if r is sufficiently small. Hence we can apply Proposition 1 to the set

$$B_r(V) = \{ w \in C_T(H^1(\Omega) \times L^2(\Omega)) \mid \operatorname{dist}(w, V) < r \}$$

and use Remark 1, the relation (21), and the homotopy invariance of the degree to complete the proof. ∎

We now come to a key statement.

PROPOSITION 3. *Let* $V \subset C_T(H^1(\Omega) \times L^2(\Omega))$ *be a bounded open set such that* F_0 *has no fixed point on the boundary of* V. *Then there exists* $r_0 > 0$ *such that to each fixed* $r \in (0, r_0]$, *there corresponds* $\varepsilon_r > 0$ *with the property that, if* $0 < \varepsilon < \varepsilon_r$, *then* F_ε *and* F^0_ε *are linearly homotopic on* $B_r(\mathscr{J} V)$. *That is, for each* $0 \leqslant \lambda \leqslant 1$, $(1-\lambda)\, F_\varepsilon + \lambda F^0_\varepsilon$ *has no fixed point on* $\partial B_r(\mathscr{J} V)$.

Proof. We first fix $r_0 > 0$ such that the operator $F_0 = J_0 f_0$ has no fixed points on the set

$$\mathbf{P}[B_r(\mathscr{J} \partial V) \cap \mathscr{J} C_T(H^1(\Omega) \times L^2(\Omega))], \qquad 0 \leqslant r \leqslant r_0.$$

The existence of such an r_0 follows from our hypotheses.

Fix $r \in (0, r_0]$. Suppose for contradiction that there exist sequences $\lambda_n \in [0, 1]$, $w_n \in \partial B_r(\mathscr{J} V) \subset Y_\varepsilon^1$, and $\varepsilon_n > 0$ such that $\lambda_n \to \lambda_0$, $\varepsilon_n \to 0$, and

$$w_n = (1 - \lambda_n)\, F_{\varepsilon_n}(w_n) + \lambda_n F_{\varepsilon_n}^0(w_n),$$

or using the notation introduced above

$$w_n = (1 - \lambda_n)\, J_{\varepsilon_n} f_{\varepsilon_n}(w_n) + \lambda_n \mathscr{J} J_0 \mathbf{P} f_{\varepsilon_n}(w_n). \tag{25}$$

Here

$$w_n(t) = \begin{pmatrix} u_n(t) \\ v_n(t) \end{pmatrix};$$

we recall that

$$f_{\varepsilon_n}(w_n)(t) = \begin{pmatrix} 0 \\ \varphi_{\varepsilon_n}(u_n)(t) \end{pmatrix}.$$

A basic observation is the following. Each u_n lies in $C_T(X_{\varepsilon_n}^1)$, and moreover the sequence $\{u_n\}$ is uniformly bounded in this space for all $\varepsilon_n > 0$. Choose $p \geqslant 2(\theta + 1)$ if $N = 1$, $p \in [2(\theta + 1), (2N + 2)/(N - 1))$ if $N \geqslant 2$. Then the set $\{u_n(t) \mid n \geqslant 1,\ 0 \leqslant t \leqslant T\}$ lies in a fixed compact subset of $L^p(Q)$, and hence there is a fixed compact $K \subset L^2(Q)$ such that

$$\varphi_{\varepsilon_n}(u_n)(t) \in K \tag{26}$$

for all $n \geqslant 1$ and all $0 \leqslant t \leqslant T$.

At this point it is convenient to introduce cutoff functions $\mathscr{X}_m \colon \mathbf{R} \to \mathbf{R}$, $m = 1, 2, \ldots$, with the following properties: $\mathscr{X}_m \in C^\infty(\mathbf{R})$, $\mathscr{X}'_m(u)$ is uniformly bounded (say by 3) with respect to m and u, $0 \leqslant \mathscr{X}_m(u) \leqslant 1$ for all $m \geqslant 1$ and all $u \in \mathbf{R}$, and

$$\mathscr{X}_m(u) = \begin{cases} 1 & |u| \leqslant m \\ 0 & |u| \geqslant m + 1. \end{cases}$$

For each $m, n \in \mathbf{N}$, we define the operator

$$\varphi_{\varepsilon_n}^m(u)(t, x, y) = \mathscr{X}_m(u(t)(x, y))\, g(t, x, \varepsilon_n y, u(t, x, y)).$$

Then $\varphi_{\varepsilon_n}^m$ maps $C_T(L^p(Q))$ to $C_T(L^2(Q))$. We will prove that

$$\varphi_{\varepsilon_n}^m(u_n) - \varphi_{\varepsilon_n}(u_n) \xrightarrow[m \to \infty]{L^2(Q)} 0, \tag{27}$$

where the convergence is uniform with respect to $n \geqslant 1$ and $t \in [0, T]$. That is, given $\delta > 0$, there exists M_1 such that, if $m \geqslant M_1$, then

$$\|\varphi^m_{\varepsilon_n}(u_n)(t) - \varphi_{\varepsilon_n}(u_n)(t)\|_{L^2(Q)} < \delta$$

for all $n \geqslant 1$ and all $0 \leqslant t \leqslant T$.

To prove (27) we write

$$\begin{aligned}\int_Q |\varphi^m_{\varepsilon_n}(u_n)(t, x, y) - \varphi_{\varepsilon_n}(u_n)(t, x, y)|^2 \\ \leqslant 2 \int_{Q^m_n(t)} |g(t, x, \varepsilon_n y, u_n(t, x, y))|^2 \, dx \, dy \\ \leqslant 2 \int_{Q^m_n(t)} \tilde{a}(1 + |u_n(t, x, y)|^{2(\theta+1)}) \, dx \, dy,\end{aligned}$$

where we have written $Q^m_n(t) = \{(x, y) \in Q \mid |u_n(t, x, y)| > m\}$ for fixed $t \in [0, T]$. Since $\{u_n(t) \mid n \geqslant 1, 0 \leqslant t \leqslant T\}$ lies in a fixed compact subset of $L^p(Q)$, we have that

$$\mu(Q^m_n(t)) \xrightarrow[m \to \infty]{} 0,$$

where the uniformity is with respect to n and to $t \in [0, T]$. Therefore since $p \geqslant 2(\theta + 1)$

$$\int_{Q^m_n(t)} |u_n(t, x, y)|^{2(\theta+1)} \, dx \, dy \xrightarrow[m \to \infty]{} 0$$

uniformly with respect to n and t. This proves (27).

We agree to write

$$f^m_{\varepsilon_n}(w) = \begin{pmatrix} 0 \\ \varphi^m_{\varepsilon_n}(u) \end{pmatrix}$$

if $w = \binom{u}{v} \in C_T(Y^1_\varepsilon)$. From (27) we have

$$\|f^m_{\varepsilon_n}(w_n)(t) - f_{\varepsilon_n}(w_n)(t)\|_{Y^1_{\varepsilon_n}} \rightrightarrows 0, \tag{28}$$

where the uniformity is with respect to $n \geqslant 1$ and $t \in [0, T]$. We further define $w^m_n = \binom{u^m_n}{v^m_n}$ by the relation

$$w^m_n = (1 - \lambda_n) J_{\varepsilon_n} f^m_{\varepsilon_n}(w_n) + \lambda_n \mathscr{J} J_0 \mathbf{P} f^m_{\varepsilon_n}(w_n). \tag{29}$$

Using the estimate (6) with $j=1$, (28) and the relation (25) for w_n, we obtain

$$\|w_n(t)-w_n^m(t)\|_{Y^1_{\varepsilon_n}} \underset{m\to\infty}{\rightrightarrows} 0, \tag{30}$$

where the convergence is uniform with respect to $n\geqslant 1$ and $t\in[0,T]$.

Let us prove that

$$\|f^m_{\varepsilon_n}(w_n)-f^m_{\varepsilon_n}(w_n^m)\|_{C_T(Y\varepsilon^1_n)} \underset{n\to\infty}{\rightrightarrows} 0, \tag{31}$$

where the uniformity is with respect to $n\geqslant 1$. In fact, relation (30) implies that there is a fixed compact set $K_1\subset L^p(Q)$ such that

$$\{w_n(t)\}_{n\geqslant 1}\cup\{w_n^m(t)\}_{n,\,m\geqslant 1}\subset K_1, \qquad 0\leqslant t\leqslant T,$$

and moreover that

$$\|u_n(t)-u_n^m(t)\|_{L_p(Q)} \underset{m\to\infty}{\rightrightarrows} 0,$$

where the uniformity is with respect to $n\geqslant 1$ and $t\in[0,T]$.

Now we have

$$\mu\{(x,y)\in Q \mid |z(x,y)|>k\} \underset{k\to\infty}{\rightrightarrows} 0, \tag{32}$$

where μ denotes Lebesgue measure on Q and the the uniformity is with respect to $z\in K_1$. We also have

$$\begin{aligned}
&\int_Q |\mathscr{X}_m(u_n(t,x,y))\,g(t,x,\varepsilon_n y,u_n(t,x,y))\\
&\qquad -\mathscr{X}_m(u_n^m(t,x,y))\,g(t,x,\varepsilon_n y,u_n^m(t,x,y))|^2\,dx\,dy\\
&\quad=\int_{Q-Q_n^m(t)} |g(t,x,\varepsilon_n y,u_n(t,x,y))-g(t,x,\varepsilon_n y,u_n^m(t,x,y))|^2\,dx\,dy\\
&\qquad+\int_{Q_n^m(t)} |\mathscr{X}_m(u_n(t,x,y))\,g(t,x,\varepsilon_n y,u_n(t,x,y))\\
&\qquad -\mathscr{X}_m(u_n^m(t,x,y))\,g(t,x,\varepsilon_n y,u_n^m(t,x,y))|^2\,dx\,dy,
\end{aligned}$$

where $Q_n^m(t) = \{(x, y) \in Q \mid |u_n(t, x, y)| > m \text{ or } |u_n^m(t, x, y)| > m\}$ for a fixed value of $t \in [0, T]$ and $n \geqslant 1$, $m \geqslant 1$. The first term on the right can be estimated by

$$\int_Q |g(t, x, \varepsilon_n y, u_n(t, x, y)) - g(t, x, \varepsilon_n y, u_n^m(t, x, y))|^2 \, dx \, dy$$

and this expression tends to zero as $m \to \infty$, uniformly in $n \geqslant 1$ and $t \in [0, T]$, because the operators φ_{ε_n} satisfy the Lipschitz condition (16) uniformly in n for elements of $C_T(L^p(Q))$ with values in K_1. Using the relation (32), we find that

$$\mu(Q_n^m(t)) \xrightarrow[m \to \infty]{} 0,$$

where the uniformity is with respect to $n \geqslant 1$ and $t \in [0, T]$. Using the estimate $|g(t, x, Y, u)| \leqslant \tilde{a}(1 + |u|^{\theta+1})$, we see that the second term on the right-hand side tends to zero as $m \to \infty$, uniformly in $n \geqslant 1$ and $t \in [0, T]$. This proves (31).

Next we remark that $f_{\varepsilon_n}^m(w_n)(t) \in Y_{\varepsilon_n}^2$ and is uniformly bounded in $n \geqslant 1$ and $t \in [0, T]$ for each *fixed* $m \geqslant 1$. Hence

$$\|f_{\varepsilon_n}^m(w_n)(t)\|_{Y_{\varepsilon_n}^2} \leqslant M(m)$$

for all $n \geqslant 1$ and $t \in [0, T]$. Using the estimate (6) with $j = 2$, we get

$$\|U_{\varepsilon_n}(t-s) f_{\varepsilon_n}^m(w_n)(s)\|_{Y_{\varepsilon_n}^2} \leqslant cM(m) \qquad (t \geqslant s). \tag{33}$$

It follows from (33) that

$$\|w_n^m(t)\|_{Y_{\varepsilon_n}^2} \leqslant c_1 M(m) \qquad (n \geqslant 1, 0 \leqslant t \leqslant T)$$

for a constant c_1 which does not depend on m. Thus one has

$$\|u_n^m(t)\|_{X_{\varepsilon_n}^2} \leqslant c_1 M(m) \tag{34}$$

$$\frac{\partial u_n^m}{\partial y}(x, 0) = 0 \qquad (x \in \Omega). \tag{35}$$

The latter relation holds because $X_{\varepsilon_n}^2 = \{u \in H^2(Q) \mid \partial u / \partial \nu = 0 \text{ in } \partial Q\}$; see [5, p. 187]. Using the inequality (9) we obtain

$$\|u_n^m(t) - Pu_n^m(t)\|_{X_{\varepsilon_n}^1} \leqslant \varepsilon_n c c_1 M(m) \tag{36}$$

for all $n \geqslant 1$ and all $0 \leqslant t \leqslant T$ (m fixed).

Now we use (36) and the Lipschitz condition satisfied by $\varphi^m_{\varepsilon_n}$ (see (16)) to conclude that

$$\|\varphi^m_{\varepsilon_n}(u^m_n)(t)-\varphi^m_{\varepsilon_n}(Pu^m_n)(t)\|_{L^2(Q)} \underset{n\to\infty}{\rightrightarrows} 0, \tag{37}$$

where the uniformity holds with respect to $t\in[0,T]$ and m is fixed. The same holds for φ_{ε_n}, that is

$$\|\varphi_{\varepsilon_n}(u^m_n)(t)-\varphi_{\varepsilon_n}(Pu^m_n)(t)\|_{L^2(Q)} \underset{n\to\infty}{\rightrightarrows} 0, \tag{38}$$

uniformly in $0\leqslant t\leqslant T$ for fixed m.

Let us now prove that, for fixed m,

$$(I-P)\,\varphi^m_{\varepsilon_n}(Pu^m_n)(t) \overset{L^2(Q)}{\underset{n\to\infty}{\rightrightarrows}} 0 \tag{39}$$

uniformly in $0\leqslant t\leqslant T$. Indeed, writing $g^m(t,x,\varepsilon y,u)=\mathscr{X}_m(u)\,g(t,x,\varepsilon y,u)$, we have

$$\left(\int_\Omega\int_0^1\left|g^m(t,x,\varepsilon_n y,Pu^m_n(t,x))-\int_0^1 g^m(t,x,\varepsilon_n z,Pu^m_n(t,x))\,dz\right|^2 dy\,dx\right)^{1/2}$$

$$\leqslant\left(\int_\Omega\int_0^1|g^m(t,x,\varepsilon_n y,Pu^m_n(t,x))-g^m(t,x,0,Pu^m_n(t,x))|^2\,dy\,dx\right)^{1/2}$$

$$+\left(\int_\Omega\int_0^1\left|\int_0^1[g^m(t,x,0,Pu^m_n(t,x))\right.\right.$$

$$\left.\left.-g^m(t,x,\varepsilon_n z,Pu^m_n(t,x))]\,dz\right|^2 dy\,dx\right)^{1/2}$$

$$\leqslant\left(\int_{\Omega_k}\int_0^1|g^m(t,x,\varepsilon_n y,Pu^m_n(t,x))-g^m(t,x,0,Pu^m_n(t,x))|^2\,dy\,dx\right)^{1/2}$$

$$+\left(\int_{\Omega-\Omega_k}\int_0^1|g^m(t,x,\varepsilon_n y,Pu^m_n(t,x))\right.$$

$$\left.-g^m(t,x,0,Pu^m_n(t,x))|^2\,dy\,dx\right)^{1/2}$$

$$+\left(\int_{\Omega_k}\int_0^1\left|\int_0^1 [\,g^m(t,x,0,Pu_n^m(t,x))\right.\right.$$

$$\left.\left.-g^m(t,x,\varepsilon_n z,Pu_n^m(t,x))]\,dz\right|^2 dy\,dx\right)^{1/2}$$

$$+\left(\int_{\Omega-\Omega_k}\int_0^1\left|\int_0^1 [\,g^m(t,x,0,Pu_n^m(t,x))\right.\right.$$

$$\left.\left.-g^m(t,x,\varepsilon_n z,Pu_n^m(t,x))]\,dz\right|^2 dy\,dx\right)^{1/2}.$$

Here $\Omega_k=\{x\in\Omega \mid$ there exists $n\geqslant 1$ and $t\in[0,T]$ such that $|Pu_n^m(t,x)| >k\}$. Since $\{Pu_n^m(t)\mid n\geqslant 1,\ 0\leqslant t\leqslant T\}$ is relatively compact in $L^p(\Omega)$, we have

$$\mu(\Omega_k)\rightrightarrows 0 \qquad (k\to\infty),$$

where the uniformity holds with respect to $n\geqslant 1$ and $t\in[0,T]$. So the first and the third terms in the final expression above tend to zero as $k\to\infty$. The second and fourth terms tend to zero as $n\to\infty$ because g is uniformly continuous in the set

$$[0,T]\times\bar{\Omega}\times[0,1]\times[-k,k].$$

This finishes the proof of (39).

Let us now rewrite the equality (25) in the form

$$\begin{aligned} w_n^m &= (1-\lambda_n)\,J_{\varepsilon_n}(f_{\varepsilon_n}(w_n)-f_{\varepsilon_n}^m(w_n))+\lambda_n\mathscr{I}J_0\mathbf{P}(f_{\varepsilon_n}(w_n)-f_{\varepsilon_n}^m(w_n))\\ &\quad+(1-\lambda_n)\,J_{\varepsilon_n}f_{\varepsilon_n}^m(w_n)+\lambda_n\mathscr{I}J_0\mathbf{P}f_{\varepsilon_n}^m(w_n)+(w_n^m-w_n). \end{aligned}$$

If we write

$$I^{nm}=J_{\varepsilon_n}(f_{\varepsilon_n}(w_n)-f_{\varepsilon_n}^m(w_n))$$

$$I_0^{nm}=\mathscr{I}J_0\mathbf{P}(f_{\varepsilon_n}(w_n)-f_{\varepsilon_n}^m(w_n)),$$

then this relation takes the form

$$\begin{aligned} w_n^m &= (1-\lambda_n)\,J_{\varepsilon_n}f_{\varepsilon_n}^m(w_n)+\lambda_n\mathscr{I}J_0\mathbf{P}f_{\varepsilon_n}^m(w_n)\\ &\quad+(1-\lambda_n)\,I^{nm}+\lambda_nI_0^{nm}+(w_n^m-w_n). \end{aligned} \tag{40}$$

From the relation (28) and the estimate (6) with $j=1$, one has

$$\|I^{nm}\|_{C_T(Y^1_{\varepsilon_n})} \rightrightarrows 0 \tag{41}$$

$$\|I_0^{nm}\|_{C_T(Y^1_{\varepsilon_n})} \rightrightarrows 0, \tag{42}$$

when $m \to \infty$, uniformly in $n \geqslant 1$. By (7) the projection $\mathbf{P}$ is bounded, and so using (41), (42), and (30) we get

$$\|\mathbf{P}I^{nm}\|_{C_T(H^1(\Omega)\times L^2(\Omega))} \rightrightarrows 0 \tag{43}$$

$$\|\mathbf{P}I_0^{nm}\|_{C_T(H^1(\Omega)\times L^2(\Omega))} \rightrightarrows 0 \tag{44}$$

$$\|(I-\mathbf{P})\, I^{nm}\|_{C_T(Y\varepsilon^1_{\varepsilon_n})} \rightrightarrows 0 \tag{45}$$

$$\|(I-\mathbf{P})\, I_0^{nm}\|_{C_T(Y^1_{\varepsilon_n})} \equiv 0 \tag{46}$$

$$\|\mathbf{P}(w_n - w_n^m)\|_{C_T(Y^1_{\varepsilon_n})} \rightrightarrows 0 \tag{47}$$

$$\|(I-\mathbf{P})(w_n - w_n^m)\|_{C_T(Y^1_{\varepsilon_n})} \rightrightarrows 0, \tag{48}$$

when $m \to \infty$, where the uniformity in all these relations is with respect to $n \geqslant 1$.

Now apply the projection $\mathbf{P}$ to the equality (40). Using the fact that $\mathbf{P}$ and $U_\varepsilon(t)$ commute, we obtain

$$\begin{aligned}\mathbf{P}w_n^m &= (1-\lambda_n)\, \mathbf{P}J_{\varepsilon_n}\mathbf{P}f^m_{\varepsilon_n}(w_n) + \lambda_n J_0 \mathbf{P} f^m_{\varepsilon_n}(w_n)\\ &\quad + (1-\lambda_n)\, \mathbf{P}I^{nm} + \lambda_n \mathbf{P}I_0^{nm} - \mathbf{P}(w_n - w_n^m).\end{aligned}$$

Since $U_\varepsilon(t)\mathbf{P} = U_0(t)\mathbf{P}$, this equality can be rewritten as

$$\begin{aligned}\mathbf{P}w_n^m &= (1-\lambda_n)\, J_0\mathbf{P}f^m_{\varepsilon_n}(w_n) + \lambda_n J_0 \mathbf{P} f^m_{\varepsilon_n}(w_n)\\ &\quad + (1-\lambda_n)\, \mathbf{P}I^{nm} + \lambda_n \mathbf{P}I_0^{nm} - \mathbf{P}(w_n - w_n^m),\end{aligned}$$

or

$$\mathbf{P}w_n^m = J_0\mathbf{P}f^m_{\varepsilon_n}(w_n) + (1-\lambda_n)\, \mathbf{P}I^{nm} + \lambda_n \mathbf{P}I_0^{nm} - \mathbf{P}(w_n - w_n^m).$$

The last equality can be transformed to read

$$\begin{aligned}\mathbf{P}w_n^m &= J_0\mathbf{P}f_{\varepsilon_n}(\mathbf{P}w_n^m) + J_0\mathbf{P}(f_{\varepsilon_n}(w_n^m) - f_{\varepsilon_n}(\mathbf{P}w_n^m))\\ &\quad + J_0\mathbf{P}(f^m_{\varepsilon_n}(w_n) - f_{\varepsilon_n}(w_n^m)) + (1-\lambda_n)\, \mathbf{P}I^{nm}\\ &\quad + \lambda_n \mathbf{P}I_0^{nm} - \mathbf{P}(w_n - w_n^m).\end{aligned} \tag{49}$$

We also apply $I-\mathbf{P}$ to the equality (40). We obtain

$$(I-\mathbf{P})\, w_n^m = (1-\lambda_n)(I-\mathbf{P})\, J_{\varepsilon_n}(I-\mathbf{P})\, f_{\varepsilon_n}^m(\mathbf{P}w_n) \\ +(1-\lambda_n)(I-\mathbf{P})\, I^{nm} + (I-\mathbf{P})(w_n^m - w_n),$$

or

$$(I-\mathbf{P})\, w_n^m = (1-\lambda_n)(I-\mathbf{P})\, J_{\varepsilon_n}(I-\mathbf{P})\, f_{\varepsilon_n}^m(w_n^m) \\ +(1-\lambda_n)(I-\mathbf{P})\, J_{\varepsilon_n}(I-\mathbf{P})(f_{\varepsilon_n}^m(w_n) - f_{\varepsilon_n}^m(w_n^m)) \\ +(1-\lambda_n)(I-\mathbf{P})\, I^{nm} + (I-\mathbf{P})(w_n^m - w_n),$$

or finally

$$(I-\mathbf{P})\, w_n^m = (1-\lambda_n)(I-\mathbf{P})\, J_{\varepsilon_n}(I-\mathbf{P})\, f_{\varepsilon_n}^m(\mathbf{P}w_n^m) \\ +(1-\lambda_n)(I-\mathbf{P})\, J_{\varepsilon_n}(I-\mathbf{P})(f_{\varepsilon_n}^m(w_n^m) - f_{\varepsilon_n}^m(\mathbf{P}w_n^m)) \\ +(1-\lambda_n)(I-\mathbf{P})\, J_{\varepsilon_n}(I-\mathbf{P})(f_{\varepsilon_n}^m(w_n) - f_{\varepsilon_n}^m(w_n^m)) \\ +(1-\lambda_n)(I-\mathbf{P})\, I^{nm} + (I-\mathbf{P})(w_n^m - w_n). \qquad (50)$$

In this way, we have replaced the relation (40) with the two relations (49), (50). These equalities hold for all $n \geqslant 1$ and $m \geqslant 1$. We pick a subsequence $\{\tilde{w}_q \mid q \geqslant 1\}$ of the set $\{w_n^m \mid n \geqslant 1,\ m \geqslant 1\}$ in the following way. First, fix $n_1 = m_1 = 1$. Supposing that m_q and n_q have been chosen, we pick $m_{q+1} > m_q$ in such a way that

$$\|\mathbf{P}I^{nm_{q+1}}\|, \quad \|\mathbf{P}I_0^{nm_{q+1}}\|, \quad \|\mathbf{P}(w_n - w_n^{m_{q+1}})\|, \\ \|(I-\mathbf{P})\, I^{nm_{q+1}}\|, \quad \|(I-\mathbf{P})(w_n - w_n^{m_{q+1}})\|, \\ \|(I-\mathbf{P})\, J_{\varepsilon_n}(I-\mathbf{P})(f_{\varepsilon_n}^{m_{q+1}}(w_n) - f_{\varepsilon_n}^{m_{q+1}}(w_n^{m_{q+1}})\|, \\ \|J_0\mathbf{P}(f_{\varepsilon_n}(w_n) - f_{\varepsilon_n}(w_n^{m_{q+1}}))\|$$

are all less than 2^{-q-1} for all $n \geqslant 1$. Here the norms are in $C_T(H^1(\Omega) \times L^2(\Omega))$ resp. $C_T(Y_\varepsilon^1)$. The existence of such a number m_{q+1} can be proved as follows. As regards the first five terms, use relations (43), (44), (47), (45), and (48), respectively. As for the sixth term, we use (31). For the seventh and final term, we use (28).

Having fixed m_{q+1}, we choose $n_{q+1} > n_q$ in such a way that

$$\|J_0\mathbf{P}(f_{\varepsilon_{n_{q+1}}}(w_{n_{q+1}}^{m_{q+1}}) - f_{\varepsilon_{n_{q+1}}}(\mathbf{P}w_{n_{q+1}}^{m_{q+1}}))\| < 2^{-q-1}, \\ \|(I-\mathbf{P})\, J_{\varepsilon_{n_{q+1}}}(I-\mathbf{P})(f_{\varepsilon_{n_{q+1}}}^{m_{q+1}}(w_{n_{q+1}}^{m_{q+1}}) - f_{\varepsilon_{n_{q+1}}}^{m_{q+1}}((\mathbf{P}w_{n_{q+1}}^{m_{q+1}}))\| < 2^{-q-1}, \\ \|(I-\mathbf{P})\, J_{\varepsilon_{n_{q+1}}}(I-\mathbf{P})\, f_{\varepsilon_{n_{q+1}}}^{m_{q+1}}(\mathbf{P}w_{n_{q+1}}^{m_{q+1}})\| < 2^{-q-1},$$

where the norms are in $C_T(H^1(\Omega)\times L^2(\Omega))$ resp. $C_T(Y^1_\varepsilon)$. The choice of such an n_{q+1} is possible because of (37) (for the first two inequalities) and (39) (for the final inequality).

Now set

$$\tilde{w}_q = w^{m_q}_{n_q} \qquad (q \geqslant 1).$$

From (50) we obtain

$$\|(I-\mathbf{P})\,\tilde{w}_q\| < 5\cdot 2^{-q-1}. \tag{51}$$

From (49) we obtain

$$\begin{aligned}\mathbf{P}\tilde{w}_q &= J_0\mathbf{P}f_{\varepsilon_{n_q}}(\mathbf{P}\tilde{w}_q) + J_0\mathbf{P}(f_{\varepsilon_{n_q}}(\tilde{w}_q) - f_{\varepsilon_{n_q}}(\mathbf{P}\tilde{w}_q))\\ &\quad + J_0\mathbf{P}(f_{\varepsilon_{n_q}}(w_{n_q}) - f_{\varepsilon_{n_q}}(\tilde{w}_q))\\ &\quad + (1-\lambda_n)\,\mathbf{P}I^{n_q m_q} - (1-\lambda_n)\,\mathbf{P}I_0^{n_q m_q} - \mathbf{P}(w_{n_q} - \tilde{w}_q).\end{aligned} \tag{52}$$

The last five terms on the right-hand side tend to zero by the construction of $\tilde{w}_q$ and the preceding remarks. The values $\{\mathbf{P}\tilde{u}_q \mid q \geqslant 1\}$ belong to a fixed compact subset of $L^p(\Omega)$. So by (29), the set $\{\mathbf{P}\tilde{w}_q \mid q\geqslant 1\}$ is relatively compact in $C_T(H^1(\Omega)\times L^2(\Omega))$, and so there is no loss of generality in assuming that $\mathbf{P}\tilde{w}_q \to w^*$ in $C_T(H^1(\Omega)\times L^2(\Omega))$.

Now use (30), (51), and the fact that $w_n \in \mathscr{J}B_r(\mathscr{J}V)$ to see that $w^* \in \partial B_r(V)$. Passing to the limit in (52), we obtain

$$w^* = F_0(w^*)$$

contrary to the choice of r. This proves Proposition 3. ∎

We can now formulate our main results.

THEOREM 1. *Suppose that the equation*

$$\frac{\partial^2 u}{\partial t^2} = \Delta_x u - \beta\frac{\partial u}{\partial t} - \alpha u + g(t, x, 0, u) \qquad (x\in\Omega)$$

with the Neumann boundary condition

$$\frac{\partial u}{\partial v} = 0 \qquad (x \in \partial\Omega)$$

has an isolated T_0-periodic solution u^0 defining an element

$$w_0 = \begin{pmatrix} u^0 \\ u^0_t \end{pmatrix} \in C_T(H^1(\Omega)\times L^2(\Omega)),$$

with topological index $\operatorname{ind}(F_0, w_0) \neq 0$. *Then for sufficiently small* $\varepsilon > 0$, *the equation* (3) *with boundary condition* (4) *has a* T_ε-*periodic solution* u^ε, *and*

$$\left\| \begin{pmatrix} u^\varepsilon \\ u_t^\varepsilon \end{pmatrix} - \mathscr{J} \begin{pmatrix} u^0 \\ u_t^0 \end{pmatrix} \right\|_{C_T(Y_\varepsilon^1)} \xrightarrow[\varepsilon \to 0]{} 0.$$

Proof. The proof follows from Propositions 1, 2, 3, and the homotopy invariance of the topological degree. ▮

THEOREM 2. *Let* $V \subset C_T(H^1(\Omega) \times L^2(\Omega))$ *be a bounded open set such that* F_0 *has no fixed points on* ∂V *and such that* $\operatorname{ind}_{C_T(H^1(\Omega) \times L^2(\Omega))}(F_0, \mathscr{U}) \neq 0$. *Then for sufficiently small* $\varepsilon \neq 0$, *there exists a* T-*periodic solution of problem* (3)–(4) *in* $C_T(Y_\varepsilon^1)$.

Proof. Use Propositions 1, 2, 3 and basic properties of the topological degree [10]. ▮

In fact, Theorem 2 can be strengthened as follows. Given $r > 0$, for sufficiently small $\varepsilon > 0$ there is a T-periodic solution of problem (3)–(4) in $B_r(\mathscr{J} V) \subset Y_\varepsilon^1$.

3. AN ILLUSTRATIVE EXAMPLE

We consider an example which is intended on the one hand to illustrate the content of our results and on the other to show more clearly that they are not comparable with those of Hale and Raugel [5].

Let us study the equation (1) with boundary condition (2) when the nonlinear term g has the form

$$g(t, x, Y, u) = \psi(u) + e(t, x, Y) + h(Y, u). \tag{53}$$

In order to satisfy the conditions (10)–(12), we require that the functions ψ, e, h be of class C^1, and that

$$|\psi'(u)| \leqslant a(1 + |u|^{\theta)}$$

$$|h_u(Y, u)| \leqslant a(1 + |u|^{\theta)}$$

$$|h_Y(Y, u)| \leqslant a(1 + |u|^{\theta+1})$$

$$|e_x(t, x, Y)| \leqslant a(1 + |u|^{\theta+1})$$

$$|e_Y(t, x, Y)| \leqslant a(1 + |u|^{\theta+1}),$$

where θ is chosen as in Section 2 and a is constant. We suppose that e is T-periodic with respect to t

$$e(t+T, x, Y) = e(t, x, Y)$$

for all values of (t, x, Y). Further, we suppose that

$$\limsup_{u \to \infty} \frac{\psi(u)}{u} \leqslant 0 \tag{54}$$

and

$$h(0, u) \equiv 0.$$

We make no further hypothesis on h if $Y \neq 0$.

We make use of the Lyapounov function introduced by Hale and Raugel in [5] to study the following one-parameter family of Neumann problems on Ω,

$$\frac{\partial^2 u}{\partial t^2} = \Delta_x u - \beta \frac{\partial u}{\partial t} - \alpha u + \lambda \psi(u) + e(t, x, 0), \tag{56}$$

with the boundary condition

$$\frac{\partial u}{\partial \nu} = 0 \qquad \text{on } \partial\Omega. \tag{57}$$

Here λ takes values in the interval $[0, 1]$. The Lyapounov function is defined for each fixed λ as follows. Let $\lambda_{1,0} > 0$ be the lowest eigenvalue of the generator A_0 of the semigroup $U_0(t)$ defined by the problem (17)–(18). Define

$$\Psi(u) = \int_0^u \psi(s)\, ds,$$

and let b be a number satisfying

$$0 < b < \min\left\{\frac{\beta}{8}, \frac{5\lambda_{1,0}}{16\beta}, \frac{\sqrt{5\lambda_{1,0}}}{8}\right\}.$$

Define the function

$$\begin{aligned} V_\lambda(u, v) = {} & \tfrac{1}{2}\, \|v\|_{L^2(\Omega)} + 2b\, \langle u, v \rangle_{L^2(\Omega)} + \tfrac{1}{2}\, \|u\|_{H^1(\Omega)} \\ & + \langle \lambda \Psi(u), 1 \rangle_{L^2(\Omega)} \end{aligned}$$

for $\binom{u}{v} \in H^1(\Omega) \times L^2(\Omega)$. Here of course 1 denotes the function which is identically 1 on Ω.

Reasoning as in [5], we can prove an a priori estimate valid for all T-periodic solutions of problems $(56)_\lambda$–(57). In other words, there exists a number $r>0$ which does not depend on λ such that the fixed points of the nonlinear operator $F_{0,\lambda}\colon C_T(H^1(\Omega)\times L^2(\Omega))\to C_T(H^1(\Omega)\times L^2(\Omega))$ defined by $(56)_\lambda$–(57) (see the lines preceding Proposition 1) all lie in the ball $B(0,r)$ centered at the origin with radius r in $C_T(H^1(\Omega)\times L^2(\Omega))$.

We agree to write F_0 for $F_{0,1}$. The family $\{F_{0,\lambda}\}$ defines a homotopy between F_0 and $F_{0,0}$ which has no fixed points on $\partial B(0,r)$. Furthermore, writing $w=\binom{u}{v}$, we have

$$F_{0,0}(w)=J_0(e),$$

where J_0 is defined in Section 2 and we have abused notation by identifying $e=e(t,x,0)$ with the element $\binom{0}{e(t)(x)}$ of $C_T(H^1(\Omega)\times L^2(\Omega))$ which it defines.

Thus $F_{0,0}$ is an operator whose range is a single point lying off $\partial B(0,r)$. Hence its topological index with respect to $B(0,r)$ is one. Hence

$$\operatorname{ind}_{C_T(H^1(\Omega)\times L^2(\Omega))}(F_0,B(0,r))=1$$

by the homotopy invariance of the index.

So, we conclude from Theorem 2 that the problem (1)–(2) with g as in (53) admits a T-periodic solution in Y^1_ε for all small $\varepsilon>0$. We note again the general nature of the function h.

REFERENCES

1. J. Arrieta, A. Carvalho, and J. Hale, A damped hyperbolic equation with critical exponent, *Comm. Partial Differential Equations* **17** (1992), 841–866.
2. I. Ciuperca, Lower semicontinuity of attractors for a reaction-diffusion equation on thin domains with varying order of thinness, Université de Paris-Sud, 1996, preprint.
3. I. Ciuperca, Reaction-diffusion equations on thin domains with varying order of thinness, *J. Differential Equations* **126** (1996), 244–291.
4. I. N. Gourova and M. I. Kamenskii, On the method of semidiscretization in periodic solutions problems for quasilinear autonomous parabolic equations, *J. Differential Equations* **32** (1996), 101–106. [in Russian]
5. J. Hale and G. Raugel, A damped hyperbolic equation on thin domains, *Trans. Amer. Math. Soc.* **329** (1992), 185–219.
6. J. Hale and G. Raugel, Reaction-diffusion equations in thin domains, *J. Math. Pures Appl.* **71** (1992), 33–95.
7. J. Hale and G. Raugel, Limits of semigroups depending on parameters, *Resenhas* **1** (1993), 1–45.
8. R. Johnson, M. I. Kamenskii, and P. Nistri, Existence of periodic solutions for an autonomous damped wave equation in a thin domains, in preparation.

9. R. Johnson, M. I. Kamenskii, and P. Nistri, On periodic solutions of a damped wave equation in a thin domain using degree theoretic methods, submitted for publication.
10. M. Krasnoselskii, P. Zabreiko, E. Pustyl'nik, and P. Sobolevski, "Integral Operators in Spaces of Summable Functions," Noordhoff, Leyden, 1976.
11. S. Krein, "Linear Differential Equations in Banach Spaces," Nauka, Moscow, 1967.
12. G. Raugel, Dynamics of partial differential equations in thin domains, *in* "Lecture Notes in Mathematics," Vol. 1609, pp. 208–315, Springer-Verlag, Berlin, 1995.